# Probability

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# Outlines

- Essential concepts in understanding probability
- Calculating probabilities using simple events
- Event relations and probability rules
- Independent events
- Conditional probabilities and Bayes’ rule

# Introduction

- The role of probability in statistics
  - **Known** population: describe the likelihood of a particular sample outcome
  - **Unknown** population: describe the properties of the population

# Concepts

**Experiment** – the process by which an observation is obtained

**Simple event** – the outcome observed on a single repetition of the experiment

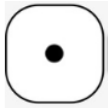
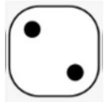
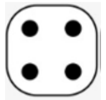
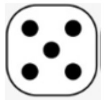
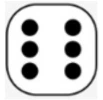
**Event** – a collection of simple events

**Mutually exclusive events** – if one event occur, the others cannot.

**Sample space** – a set of all possible simple events

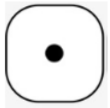
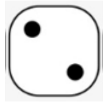
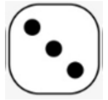
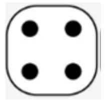
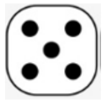
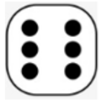
# Example 1

- Experiment: Roll the dice 100 times and observe the results.

- Simple events:  ,  ,  ,  ,  , 

- Event: even numbers are observed.

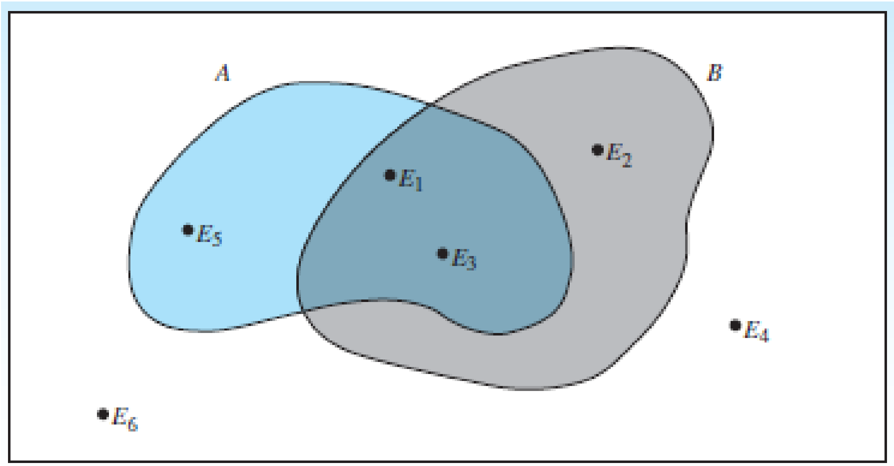
- Mutually exclusive events: all simple events are mutually exclusive.

- Sample space:  +  +  +  +  + 

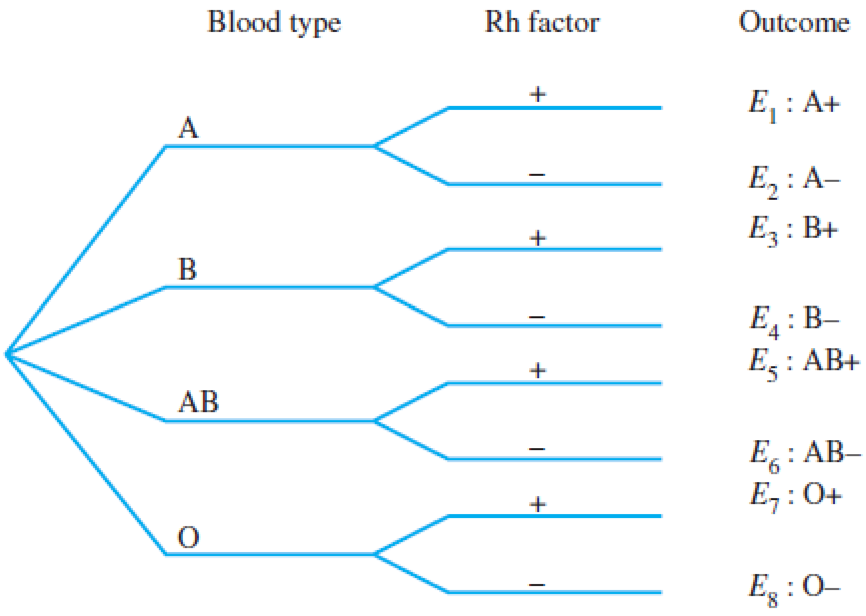
# Example 2

- **Experiment** – collect the age of 100 random males and 100 random females and put them in bins of U16, 17-50, 51-65, over 66
- **Simple events –** Assuming none of the 200 people was over 66. There was at least one observation of male and of female in each age group. Simple events are:
  - Male U16, Male 17-50, and Male 51-65
  - Female U16, Female 17-50, and Female 51-65
- **Events**
  - Event A: a person under 50 is picked.
  - Event B: a male is picked
  - Event C: a female is picked
- **Mutually exclusive events**
  - Events B and C are mutually exclusive.
  - Events A and B (or A and C) are not mutually exclusive.
- **Sample space** – comprised by all simple events

# Describing sample space




Venn diagram




Tree diagram

# Calculating probabilities using simple events

- Relative frequency, $\frac{\text{Frequency}}{n}$
- Probability of an event A, $P(A) = \lim_{n \to \infty} \frac{\text{Frequency}}{n}$
- It also equals the sum of probability of all simple events contained in A.
  - List all simple events in the sample space, i.e. the probability of all simple events considered MUST sum to 1.
  - Assign an appropriate probability for each simple event
  - Determine simple events resulting in the event of interest
  - Sum the probabilities of those simple events

# Example 3

- Event A: An observation of calcium between 400mg and 1000mg
- What are the simple events contained in A?
- What is the probability of event A?

| *Calcium(mg)* | *Frequency* | *Relative Frequency* |
|---|---|---|
| (0 - 200] | 65 | 0.09 |
| (200 - 400] | 174 | 0.24 |
| (400 - 600] | 178 | 0.24 |
| (600 - 800] | 123 | 0.17 |
| (800 - 1000] | 82 | 0.11 |
| (1000 - 1200] | 52 | 0.07 |
| (1200 - 1400] | 28 | 0.04 |
| (1400 - 1600] | 16 | 0.02 |
| (1600 - 1800] | 7 | 0.01 |
| (1800 - 2000] | 7 | 0.01 |
| (2000 - 2200] | 3 | 0.00 |
| (2200 - 2400] | 0 | 0.00 |
| (2400 - 2600] | 1 | 0.00 |
| (2600 - 2800] | 0 | 0.00 |
| (2800 - 3000] | 1 | 0.00 |
| **Total** | **737** | **1.00** |

# Example 2 – cont.

- **Experiment** – collect the age of 100 random males and 100 random females and put them in bins of U16, 17-50, and 51-65 (assuming no one above 65 was observed)

- **Events:**
  - Event A: a person under 50 is picked.
  - Event B: a male is picked
  - Event C: a female is picked

| | Male | Female |
|---|---|---|
| <16 | 30 | 18 |
| 17-50 | 50 | 65 |
| 51-65 | 20 | 27 |
| | **100** | **100** |

- **Questions**:
  - Draw a tree diagram of the sample space
  - What are the simple events contained in A, B, and C?
  - What is the probability of event A?

# A review of useful counting rules

- Counting rules are helpful in identifying the number of simple events N in experiments, especially when N is *large*.

- **The mn-Rule**

  If an experiment is done in k stages with $n_k$ ways to accomplish a stage k, the number of ways to accomplish the experiment, i.e. the number of simple events, is $n_1 n_2 n_3 \ldots n_k$.

- Examples:
  - Roll three 6-face dices, the total number of results is 6 x 6 x 6 = 216
  - The total number possible combinations of male and female in 4 age groups are 2 x 4 = 8.
  - There are 3 books A, B, C and 2 slots. The total number of ways to organize the books is 3x2=6

# A review of useful counting rules

- **A counting rule for permutations (order of objects is important)**

  The total number of ways to arrange n distinct objects, taking them r at a time is

  $P_r^n = \frac{n!}{(n-r)!}$ where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$

- Examples:
  - The total number of ways to arrange 5 different books is

$$P_5^5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5 * 4 * 3 * 2 * 1 = 120$$

  - The total number of ways to select 5 people from 8 people (and order is important) is

$$P_5^8 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 8 * 7 * 6 * 5 * 4 = 6720$$

# A review of useful counting rules

- **A counting rule for combinations (order of objects is NOT important)**

  The total number of ways to combine n distinct objects, taking them r at a time is

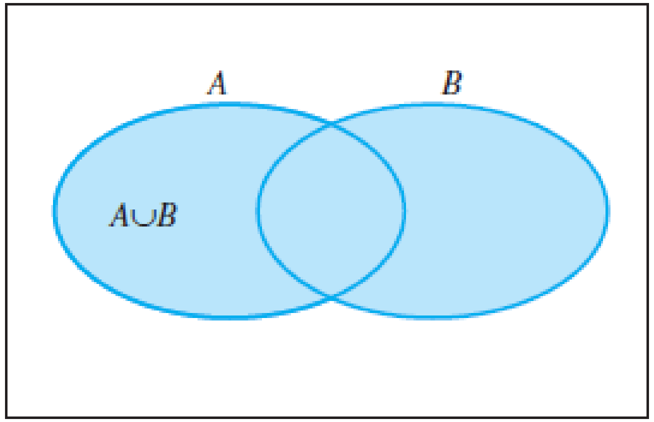
  $C_r^n = \frac{n!}{r!(n-r)!}$ where $n! = n(n-1)(n-2)\ldots(3)(2)(1)$

- Examples:
  - The total number of ways to pick 3 books out of 5 different books is

$$C_5^8 = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\,2!} = 10$$

  - How many ways are there to pick 10 nurses out of 90 for a study in determining the attitudes of nurses toward various admin procedures? Is the order of selecting the nurses important?

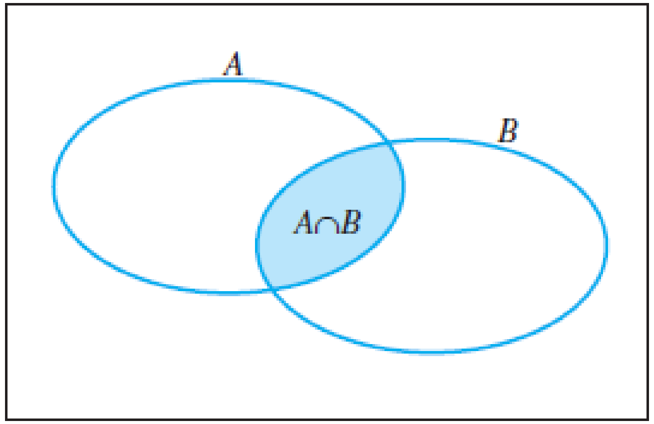
# Event Relations and Probability Rules




**Union of A and B**: either A or B or both occur

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

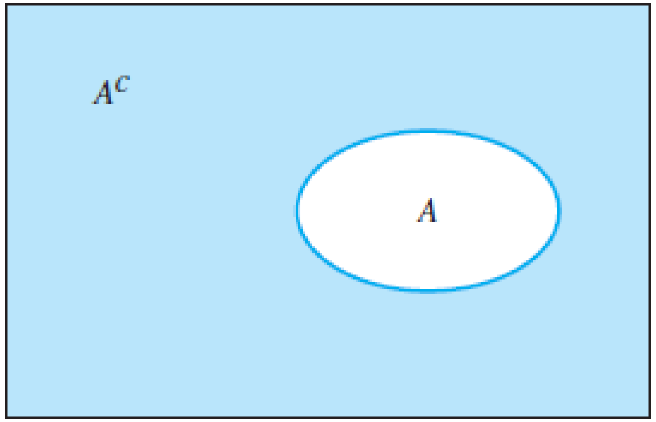
**The Addition Rule**




**Intersection of A and B**: both A and B occur

$$P(A \cap B) = P(A)P(B|A)$$
$$P(A \cap B) = P(B)P(A|B)$$

**The Multiplication Rule**




**Complement of A**: A does not occur

$$P(A^C) = 1 - P(A)$$
$$A \cup A^C = S$$

**The Rule for Complements**

# Example 4

- Toss 2 fair coins and record the outcomes. Below are the events of interest
  - A: Observe at least 1 head
  - B: Observe 2 different faces

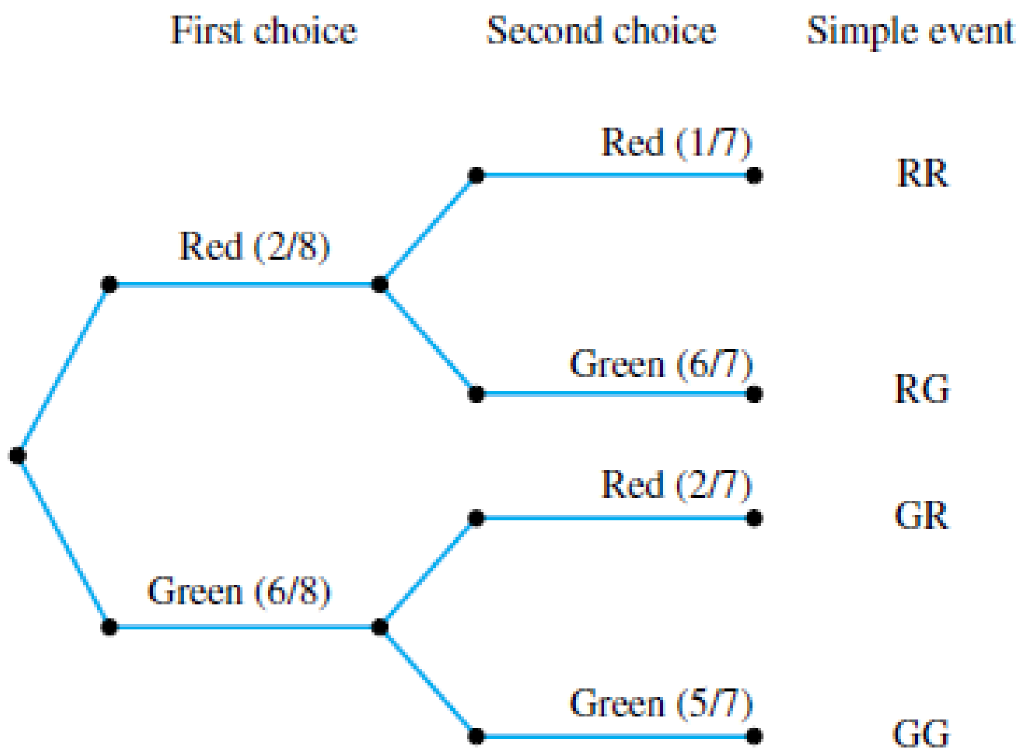
- Simple events (can be from a tree diagram)
  - E1: HH, P(E1) = ¼ E2: HT, P(E2) = ¼
  - E3: TH, P(E3) = ¼ E4: TT , P(E4) = ¼

- A = {E1, E2, E3}, P(A) = ¾ B = {E2, E3}, P(B) = 2/4

- A ∪ B = {E1, E2, E3}, P(A ∪ B) = ¾
- A ∩ B = {E2, E3}, P(A ∩ B) = ½
- $A^c$={E4}, $P(A^c) = \frac{1}{4}$

# Example 5

- There are 8 toys in a container – 2 red and 6 green. Pick random 2 toys.
- Event A: What is the probability of picking up 2 red toys?




$$P(A) = P(Red\ on\ 1st)P(Red\ on\ 2nd \mid Red\ on\ 1st)$$

$$P(A) = \left(\frac{2}{8}\right)\left(\frac{1}{7}\right) = 1/28$$

# Example 2 - cont.

- **Events:**
  - Event A: a person under 50 is picked.
  - Event B: a male is picked
  - Event C: a female is picked

| | Male | Female |
|---|---|---|
| <16 | 30 | 18 |
| 17-50 | 50 | 65 |
| 51-65 | 20 | 27 |
| | **100** | **100** |

- What is the probability of event A?
- What is the probability of event B?
- What is the probability of a male under 50 (A∩B)?
- What is the probability of a person under 50 or a female (A ∪ C)
- What is the probability of a person over 50 ($A^c$)?

# Independent events

- Event A and event B are **independent** if and only if

  $P(A|B) = P(A)$ or $P(A \cap B) = P(A)P(B)$

- Extension of multiplication rules for three independent events

  $$P(A \cap B \cap C) = P(A)P(B)P(C)$$

- **Example:** Roll 3 dices and observe the outcome. What is the probability of having 3 ⚅?

# Checking independent events

- Roll a single dice and consider the following events
  - Event E: getting an even number
  - Event T: getting a number divisible by three

- Questions:
  - What is the probability of E?
  - What is the probability of getting an even number (Event E) if you are told that the number was also divisible by three (Event T)?
  - Does knowing that the number is divisible by 3 (Event T) change the probability that the number was even (Event E)?

  **Are Event E and Event T independent?**

Source: https://www.siyavula.com/read/maths/grade-11/probability/10-probability-02

# Independent Events vs Mutual Exclusive Events

- Mutually exclusive events
  - Cannot both happen, e.g. head and tail cannot both happen in a coin toss
  - If A happened, B cannot happen, P(B|A) = 0
  - Therefore mutually exclusive events are *dependent.*
  - $P(A \cap B) = 0$, $P(A \cup B) = P(A) + P(B)$

- Independent events
  - $P(A \cap B) = P(A)P(B)$, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

- **Example**: What is the probability of drawing an ace and a 10 from a deck of 52 cards?

# Conditional Probabilities

- Conditional probability of an event B given that event A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad \text{if } P(A) \neq 0$$

- Examples:
  - What is the probability of a person <16 (Event $B_1$) given that the person is a male (Event A)?
  - What is the probability of a person a male (Event A) given that he is <16 (Event $B_1$)?
  - Is $P(A|B_1) = P(B_1|A)$?

| | Male (A) | Female ($A^c$) | |
|---|---|---|---|
| <16 ($B_1$) | 30 | 18 | **48** |
| 17-50 ($B_2$) | 50 | 55 | **105** |
| 51-65 ($B_3$) | 20 | 27 | **47** |
| | **100** | **100** | **200** |

| | Male (A) | Female ($A^c$) | |
|---|---|---|---|
| <16 ($B_1$) | 0.15 | 0.09 | **0.24** |
| 17-50 ($B_2$) | 0.25 | 0.275 | **0.525** |
| 51-65 ($B_3$) | 0.1 | 0.135 | **0.235** |
| | **0.5** | **0.5** | **1** |

# Bayes' Rule

- Bayes' rule of conditional probability

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)} \qquad \text{for } i = 1, 2, \ldots, k$$

- $B_1, \ldots, B_j$ must be mutually exclusive and $\sum_{j=1}^{k} P(B_j) = 1$
- Back to the example in the previous slide

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A)}$$

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$P(B_1|A) = \frac{0.24*(30/48)}{0.24*(30/48)+0.525*(50/105)+0.235*(20/47)} = 0.3$$

| | Male (A) | Female ($A^c$) | |
|---|---|---|---|
| <16 ($B_1$) | 30 | 18 | **48** |
| 17-50 ($B_2$) | 50 | 55 | **105** |
| 51-65 ($B_3$) | 20 | 27 | **47** |
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| | Male (A) | Female ($A^c$) | |
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| | **0.5** | **0.5** | **1** |

# Bayes' Rule

$$P(B_i|A) = P(B_i) * \frac{P(A|B_i)}{P(A)}$$ for k = 1, 2, …, k

- $P(B_i)$ is **prior probability** – without knowledge of the condition A. Can be approximated as 1/k if unknown.

- $P(B_i|A)$ is **posterior probability** – the updated version of the prior probability after observing information of the condition A in the sample.

# Bayes' Rule

$$P(B_i|A) = P(B_i) * \frac{P(A|B_i)}{P(A)}$$

- Example:
  - 60% of businesses that replaced their CEO last year has share price increased by >5%.
  - 35% of businesses that replaced their CEO last year doesn't have share price increased by >5%.
  - Last year data showed that the probability of share price increased by >5% is 4%.
  - What is the probability of a company's share price increased by >5% given it replaced the CEO?
- Solution hints
  - Event A: A CEO being replaced
  - Event $B_1$: A business has share price increased by >5%.
  - What is the prior probability of a company having share price increased by >5%?
  - What is the probability of a CEO being replaced?
  - What is the probability of a CEO being replaced given that the share price increased by >5%?

Source: https://corporatefinanceinstitute.com/resources/knowledge/other/bayes-theorem/