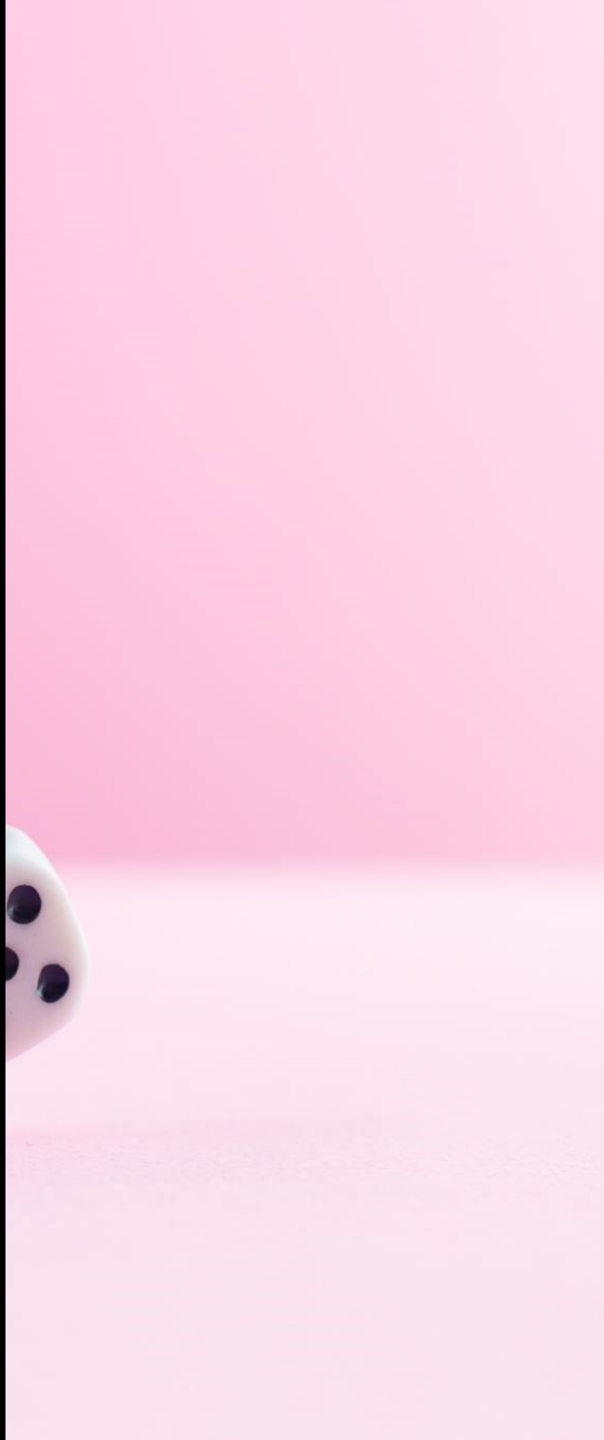
Probability

Le Thai Ha



Outlines

- Essential concepts in understanding probability
- Calculating probabilities using simple events
- Event relations and probability rules
- Independent events
- Conditional probabilities and Bayes’ rule

Introduction

- The role of probability in statistics
  - **Known** population: describe the likelihood of a particular sample outcome
  - **Unknown** population: describe the properties of the population

Concepts

Experiment – the process by which an observation is obtained

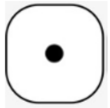
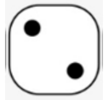
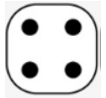
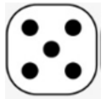
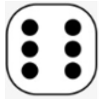
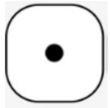
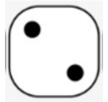
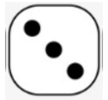
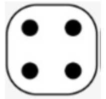
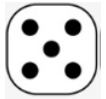
Simple event – the outcome observed on a single repetition of the experiment

Event – a collection of simple events

Mutually exclusive events – if one event occur, the others cannot.

Sample space – a set of all possible simple events

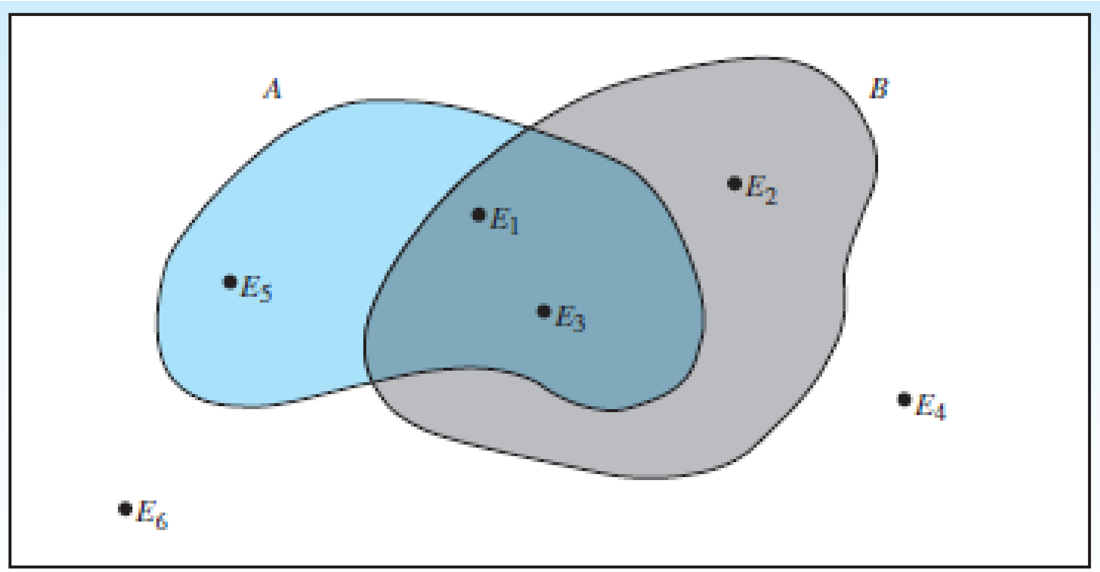
Example 1

- Experiment: Roll the dice 100 times and observe the results.
- Simple events:  ,  ,  ,  ,  , 
- Event: even numbers are observed.
- Mutually exclusive events: all simple events are mutually exclusive.
- Sample space:  +  +  +  +  + 

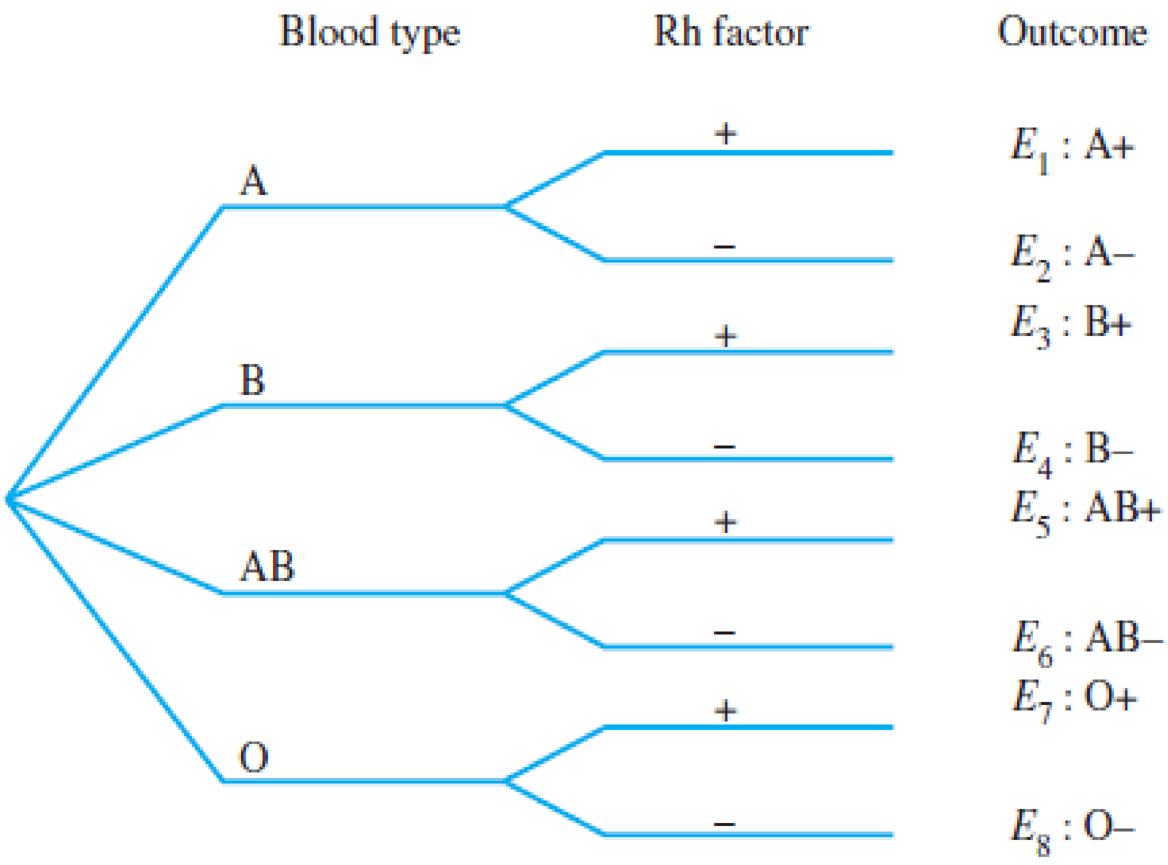
Example 2

- **Experiment** – collect the age of 100 random males and 100 random females and put them in bins of U16, 17-50, 51-65, over 66
- **Simple events –** Assuming none of the 200 people was over 66. There was at least one observation of male and of female in each age group. Simple events are:
  - Male U16, Male 17-50, and Male 51-65
  - Female U16, Female 17-50, and Female 51-65
- **Events**
  - Event A: a person under 50 is picked.
  - Event B: a male is picked
  - Event C: a female is picked
- **Mutually exclusive events**
  - Events B and C are mutually exclusive.
  - Events A and B (or A and C) are not mutually exclusive.
- **Sample space** – comprised by all simple events

Describing sample space



Venn diagram



Tree diagram

Calculating probabilities using simple events

- Relative frequency, $\frac{\text{Frequency}}{n}$
- Probability of an event A, $P(A) = \lim_{n\to\infty} \frac{\text{Frequency}}{n}$
- It also equals the sum of probability of all simple events contained in A.
  - List all simple events in the sample space, i.e. the probability of all simple events considered MUST sum to 1.
  - Assign an appropriate probability for each simple event
  - Determine simple events resulting in the event of interest
  - Sum the probabilities of those simple events

Example 3

- Event A: An observation of calcium between 400mg and 1000mg
- What are the simple events contained in A?
- What is the probability of event A?

Calcium(mg)	*Frequency*	*Relative Frequency*
(0 - 200]	65	0.09
(200 - 400]	174	0.24
(400 - 600]	178	0.24
(600 - 800]	123	0.17
(800 - 1000]	82	0.11
(1000 - 1200]	52	0.07
(1200 - 1400]	28	0.04
(1400 - 1600]	16	0.02
(1600 - 1800]	7	0.01
(1800 - 2000]	7	0.01
(2000 - 2200]	3	0.00
(2200 - 2400]	0	0.00
(2400 - 2600]	1	0.00
(2600 - 2800]	0	0.00
(2800 - 3000]	1	0.00
Total	**737**	**1.00**

Example 2 – cont.

- **Experiment** – collect the age of 100 random males and 100 random females and put them in bins of U16, 17-50, and 51-65 (assuming no one above 65 was observed)

- **Events:**
  - Event A: a person under 50 is picked.
  - Event B: a male is picked
  - Event C: a female is picked

	Male	Female
<16	30	18
17-50	50	**55**
51-65	20	27
	100	**100**

- **Questions**:
  - Draw a tree diagram of the sample space
  - What are the simple events contained in A, B, and C?
  - What is the probability of event A?

A review of useful counting rules

- Counting rules are helpful in identifying the number of simple events N in experiments, especially when N is *large*.

- **The mn-Rule**

  If an experiment is done in k stages with n_k ways to accomplish a stage k, the number of ways to accomplish the experiment, i.e. the number of simple events, is $n_1 n_2 n_3 \ldots n_k$.

- Examples:
  - Roll three 6-face dices, the total number of results is 6 x 6 x 6 = 216
  - The total number possible combinations of male and female in 4 age groups are 2x4=8.
  - There are 3 books A, B, C and 2 slots. The total number of ways to organize the books is 3x2=6

A review of useful counting rules

- **A counting rule for permutations (chỉnh hợp) (order of objects is important)**

  The total number of ways to arrange n distinct objects, taking them r at a time is

  $P_r^n = \frac{n!}{(n-r)!}$ where $n! = n(n-1)(n-2)\dots(3)(2)(1)$

- Examples:
  - The total number of ways to arrange 5 different books is

$$P_5^5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5*4*3*2*1 = 120$$

  - The total number of ways to select 5 people from 8 people (and order is important) is

$$P_5^8 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 8*7*6*5*4 = 6720$$

A review of useful counting rules

- **A counting rule for combinations (tổ hợp) (order of objects is NOT important)**

  The total number of ways to combine n distinct objects, taking them r at a time is

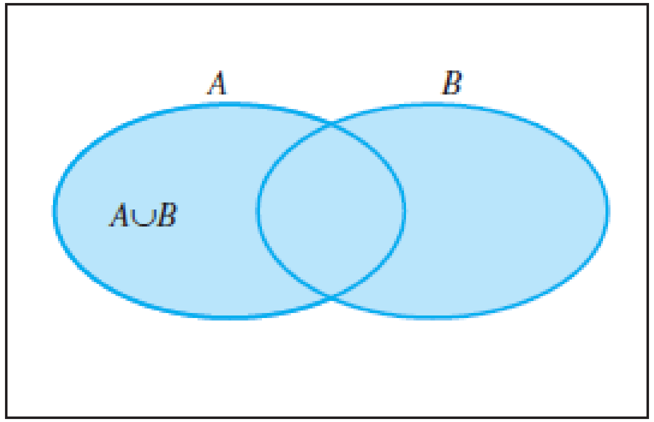
  $C_r^n = \frac{n!}{r!(n-r)!}$ where $n! = n(n-1)(n-2)\dots(3)(2)(1)$

- Examples:
  - The total number of ways to pick 3 books out of 5 different books is

$$C_5^8 = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\,2!} = 10$$

  - How many ways are there to pick 10 nurses out of 90 for a study in determining the attitudes of nurses toward various admin procedures? Is the order of selecting the nurses important?

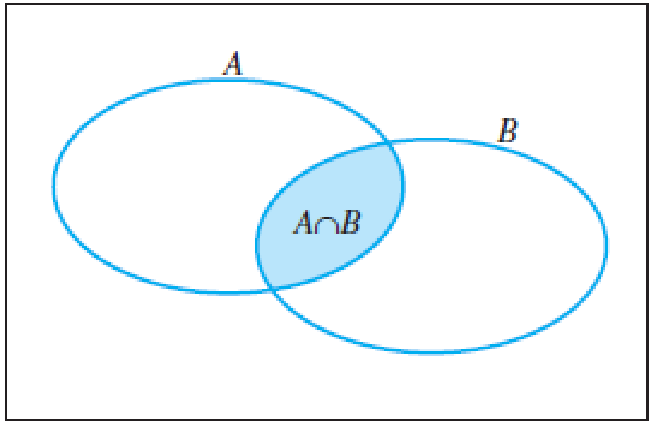
Event Relations and Probability Rules



Union of A and B: either A or B or both occur

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

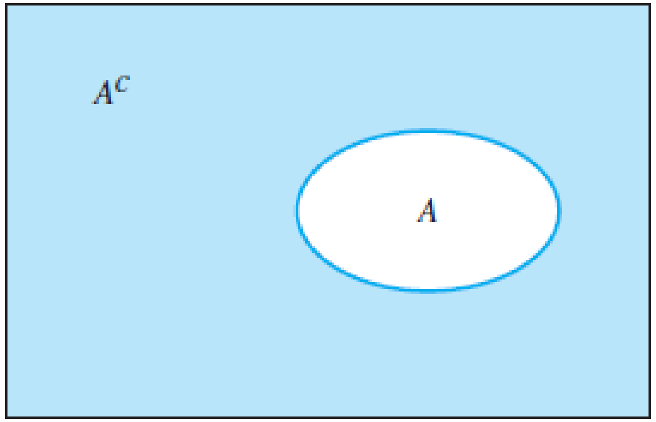
The Addition Rule



Intersection of A and B: both A and B occur

$$P(A \cap B) = P(A)P(B|A)$$
$$P(A \cap B) = P(B)P(A|B)$$

The Multiplication Rule



Complement of A: A does not occur

$$P(A^C) = 1 - P(A)$$
$$A \cup A^C = S$$

The Rule for Complements

Example 4

- Toss 2 fair coins and record the outcomes. Below are the events of interest
  - A: Observe at least 1 head
  - B: Observe 2 different faces

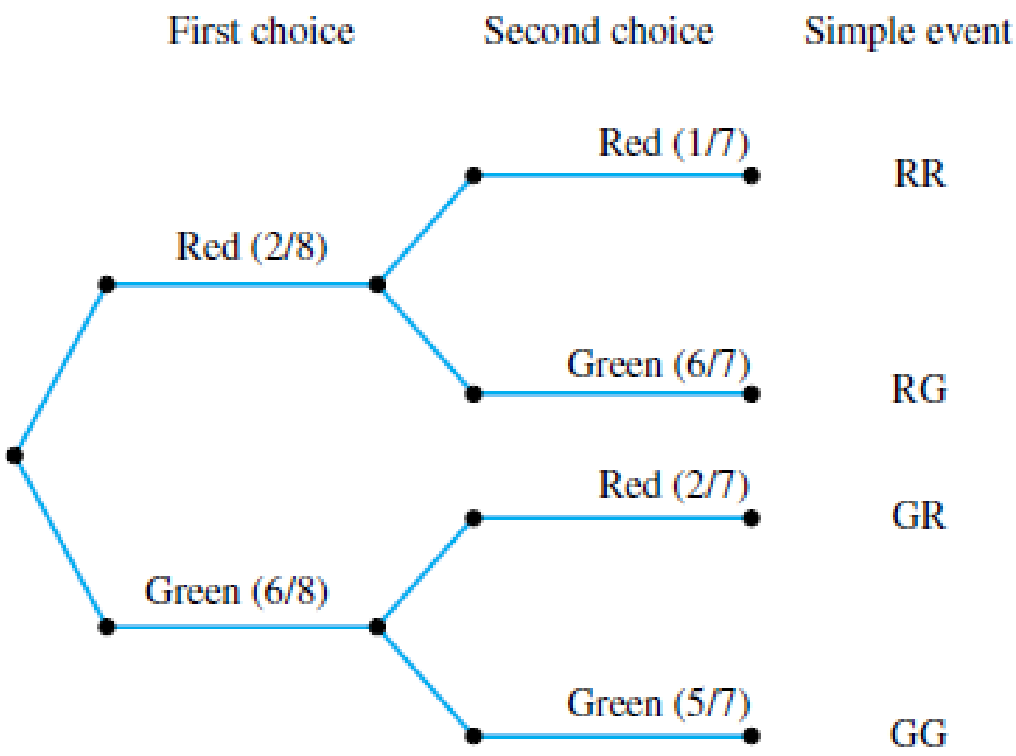
- Simple events (can be from a tree diagram)
  - E1: HH, P(E1) = ¼ E2: HT, P(E2) = ¼
  - E3: TH, P(E3) = ¼ E4: TT , P(E4) = ¼

- A = {E1, E2, E3}, P(A) = ¾ B = {E2, E3}, P(B) = 2/4

- A ∪ B = {E1, E2, E3}, P(A ∪ B) = ¾
- A ∩ B = {E2, E3}, P(A ∩ B) = ½
- A^c={E4}, $P(A^c) = \frac{1}{4}$

Example 5

- There are 8 toys in a container – 2 red and 6 green. Pick random 2 toys.
- Event A: What is the probability of picking up 2 red toys?



$$P(A) = P(Red\ on\ 1st)P(Red\ on\ 2nd \mid Red\ on\ 1st)$$

$$P(A) = \left(\frac{2}{8}\right)\left(\frac{1}{7}\right) = 1/28$$

Example 2 - cont.

- **Events:**
  - Event A: a person under 50 is picked.
  - Event B: a male is picked
  - Event C: a female is picked

	Male	Female
<16	30	18
17-50	50	**55**
51-65	20	27
	100	**100**

- What is the probability of event A?
- What is the probability of event B?
- What is the probability of a male under 50 (A∩B)?
- What is the probability of a person under 50 or a female (A ∪ C)
- What is the probability of a person over 50 (A^c)?

Independent events

- Event A and event B are **independent** if and only if

  $P(A|B) = P(A)$ or $P(A \cap B) = P(A)P(B)$

- Extension of multiplication rules for three independent events

  $$P(A \cap B \cap C) = P(A)P(B)P(C)$$

- **Example:** Roll 3 dices and observe the outcome. What is the probability of having 3 ⚅?

Checking independent events

- Roll a single dice and consider the following events
  - Event E: getting an even number
  - Event T: getting a number divisible by three

- Questions:
  - What is the probability of E?
  - What is the probability of getting an even number (Event E) if you are told that the number was also divisible by three (Event T)?
  - Does knowing that the number is divisible by 3 (Event T) change the probability that the number was even (Event E)?

Are Event E and Event T independent?

Source: https://www.siyavula.com/read/maths/grade-11/probability/10-probability-02

Independent Events vs Mutual Exclusive Events

- Mutually exclusive events
  - Cannot both happen, e.g. head and tail cannot both happen in a coin toss
  - If A happened, B cannot happen, P(B|A) = 0
  - Therefore mutually exclusive events are *dependent.*
  - $P(A \cap B) = 0$, $P(A \cup B) = P(A) + P(B)$

- Independent events
  - $P(A \cap B) = P(A)P(B)$, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

- **Example**: What is the probability of drawing an ace and a 10 from a deck of 52 cards?

Conditional Probabilities

- Conditional probability of an event B given that event A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \qquad \text{if } P(A) \neq 0$$

- Examples:
  - What is the probability of a person <16 (Event B_1) given that the person is a male (Event A)?
  - What is the probability of a person a male (Event A) given that he is <16 (Event B_1)?
  - Is $P(A|B_1) = P(B_1|A)$?

	Male (A)	Female (A^c)	
<16 (B_1)	30	18	**48**
17-50 (B_2)	50	55	**105**
51-65 (B_3)	20	27	**47**
	100	**100**	**200**

	Male (A)	Female (A^c)	
<16 (B_1)	0.15	0.09	**0.24**
17-50 (B_2)	0.25	0.275	**0.525**
51-65 (B_3)	0.1	0.135	**0.235**
	0.5	**0.5**	**1**

Bayes' Rule

- Bayes' rule of conditional probability

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)} \qquad \text{for } i = 1, 2, \dots, k$$

- B_1, …, B_j must be mutually exclusive and $\sum_{j=1}^{k} P(B_j) = 1$
- Back to the example in the previous slide

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A)}$$

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$P(B_1|A) = \frac{0.24*(30/48)}{0.24*(30/48)+0.525*(50/105)+0.235*(20/47)} = 0.3$$

$$P(B_1|A) = \frac{P(B_1 \cap A)}{P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A)}$$

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

Bayes' Rule

$$P(B_i|A) = P(B_i) * \frac{P(A|B_i)}{P(A)}$$ for k = 1, 2, …, k

- $P(B_i)$ is **prior probability** – without knowledge of the condition A. Can be approximated as 1/k if unknown.

- $P(B_i|A)$ is **posterior probability** – the updated version of the prior probability after observing information of the condition A in the sample.

Bayes’ Rule (revised example)

$$P(B_i|A) = P(B_i) * \frac{P(A|B_i)}{P(A)}$$

- Example:
  - 60% of businesses that has share price increased by >5% replaced their CEO last year.
  - 35% of businesses that doesn’t have share price increased by >5% replaced their CEO last year.
  - Last year data showed that the probability of share price increased by >5% is 4%.
  - What is the probability that the shares of a company that replaced the CEO will increase by >5%?
- Solution hints
  - Event A: A business has share price increased by >5%.
  - Event B_1: A CEO being replaced.
  - What is the prior probability of a company having share price increased by >5%?
  - What is the probability of a CEO being replaced?
  - What is the probability of a company’s share price increased by >5% given the CEO was replaced?

Source: https://corporatefinanceinstitute.com/resources/knowledge/other/bayes-theorem/